PRCSG 2022

Disk Potential Functions for Polygon Spaces

(Joint with Siu-Cheong Lau & Xiao Zheng in progress)

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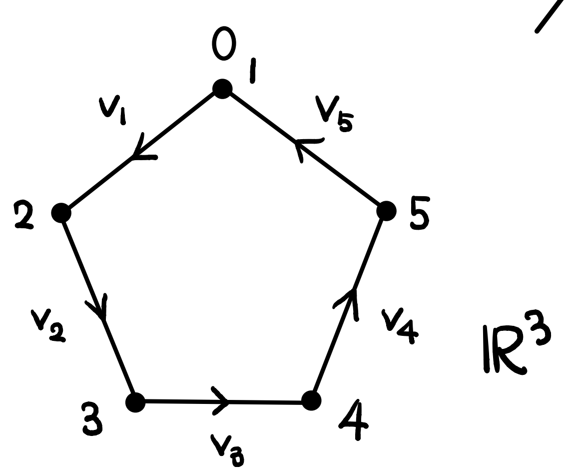
Polygon spaces



Fix $\vec{r} = (r_1, r_2, \cdots, r_{n+3}) \in \mathbb{R}^{n+3}_{>0}$ (the length of edges)

$SO(3) := SO(3;\mathbb{R}) \curvearrowright \mathbb{R}^3$ linear

$SO(3) \curvearrowright S^2(r) := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$

Polygon space

$$\mathcal{M}_{\vec{r}} := \left\{ \vec{v} = (v_1, v_2, \cdots, v_{n+3}) \in \prod_{i=1}^{n+3} S^2(r_i) : \sum_{i=1}^{n+3} v_i = 0 \right\} \Big/ SO(3) \text{ (diagonal action)}$$

Some facts

- $\mathcal{M}_{\vec{r}} \neq \phi$ if $r_i < r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_{n+3}$ $(i = 1, 2, \cdots, n+3)$.
- $\mathcal{M}_{\vec{r}} \simeq (\mathbb{C}P^1)^{n+3} /\!/ PGL(2, \mathbb{C})$ projective variety
- $\vec{r}$ is called generic if $\exists$ no $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_{n+3}) \in \{\pm 1\}^{n+3}$ s.t. $\sum_{i=1}^{n+3} \varepsilon_i r_i = 0$

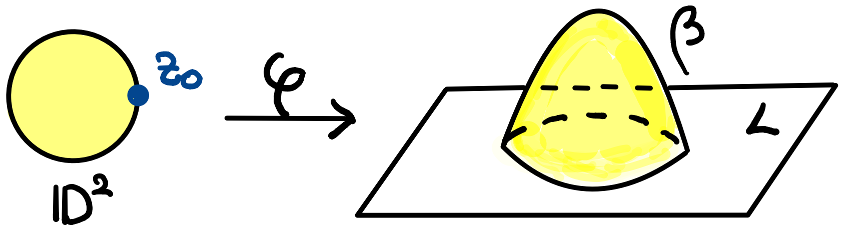
  $\mathcal{M}_{\vec{r}}$ is smooth $\Leftrightarrow$ $\vec{r}$ is generic
- $\mathcal{M}_{\vec{r}}$ is Kähler, $\dim_{\mathbb{C}} \mathcal{M}_{\vec{r}} = n$.

Disk potential functions

(Fukaya – Oh – Ohta – Ono)

- L: a Lagrangian torus of (X, ω), $\left(\omega|_{TL\times TL} = 0,\ \dim_{\mathbb{R}} L = \dim_{\mathbb{R}} X/2\right)$
  $J \curvearrowright TM$, $J^2 = -\mathrm{Id}$, $\beta \in \pi_2(X, L)$

$\mathcal{M}_1(\beta) := \overline{\Big\{}$



$\overline{\Big| \ d\varphi \circ j = J \circ d\varphi \Big\}} \Big/ \sim$

$$\text{vir.}\dim_{\mathbb{R}} \mathcal{M}_1(\beta) = \dim_{\mathbb{R}} L + \mu_L(\beta) - 3 + 1$$

- Assume that $\nexists$ stable map bounded by L of $\mu_L(\beta) \le 0$ $\left(\Rightarrow \partial \mathcal{M}_1(\beta) = \phi\right)$

- Counting invariants (a.k.a open Gromov – Witten invariants)
  $n_\beta :=$ the degree of $ev_0 : \mathcal{M}_1(\beta) \to L$ $\left(\varphi \longmapsto \varphi(z_0)\right)$
  (n_β can be nonzero only when $\dim_{\mathbb{R}} \mathcal{M}_1(\beta) = \dim_{\mathbb{R}} L$ & $\partial \mathcal{M}_1(\beta) = \phi$)

- $\mathcal{L} \to L$ trivial line bundle
  $\{\nabla \mid \nabla \text{ is a flat } \mathbb{C}^*\text{-connection on } \mathcal{L}\}/\sim$ serves a complex mirror chart.

Disk potential functions (cont.)

- $L \simeq T^n$. Choose oriented loops $\theta_1, \theta_2, \cdots, \theta_n$ s.t.

  $\pi_1(L) \simeq \mathbb{Z}^n \simeq \mathbb{Z}\langle \theta_1, \theta_2, \cdots, \theta_n \rangle$

$\{\nabla \mid \nabla \text{ is a flat } \mathbb{C}^*\text{-connection on } \mathcal{L}\}/\sim \;\simeq (\mathbb{C}^*)^n \qquad \nabla \longmapsto (x_i := \mathrm{hol}_\nabla(\theta_i))$

Definition (Disk potential function)

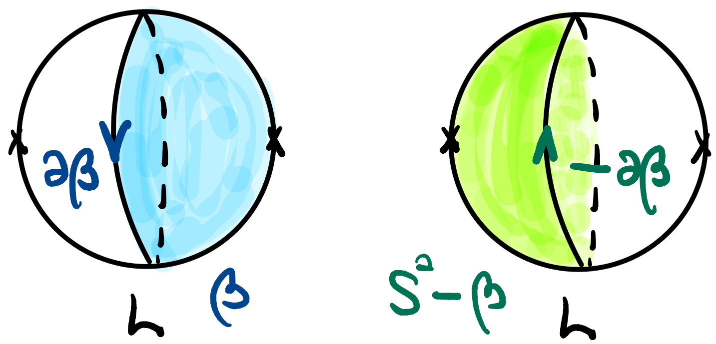
$$W_L(x) := \sum_\beta n_\beta \cdot x^{\partial\beta} \qquad W_L : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$$

e.g.> $(X = \mathbb{CP}^1, \omega_{FS})$, $L = $ (equator).

$\pi_2(X, L) \simeq \mathbb{Z}^2 \simeq \mathbb{Z}\langle \beta, S^2 - \beta \rangle$

Take $\theta := \partial\beta$ and set $x := \mathrm{hol}_\nabla(\theta) \in \mathbb{C}^*$

$$W_L(x) = n_\beta x^{\partial\beta} + n_{S^2-\beta}\, x^{\partial(S^2-\beta)} = x + \frac{1}{x}$$



- More generally, may have mutiple Lagrangian tori $\rightsquigarrow$ multiple $(\mathbb{C}^*)^n$ charts.

  Charts are glued to produce a "better" mirror.

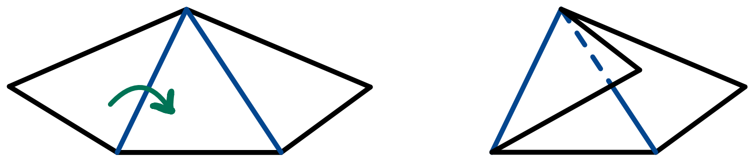
Bending systems

Want to construct Lagrangian tori in $\mathcal{M}_{\vec{r}}$ (polygon space)

(Kapovich – Millson)

- $\mathcal{M}_{\vec{r}}$ has a Hamiltonian torus action (defined on its dense open subset)

  For $\vec{r} = (r_1, r_2, \cdots, r_{n+3}) \in \mathbb{R}^{n+3}_{>0}$, we take a triangulation of $(n+3)$-gon



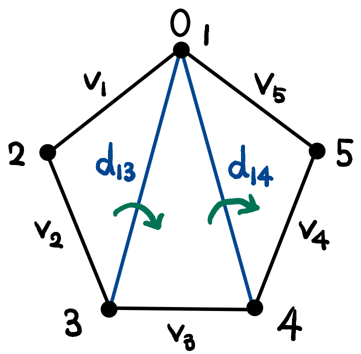
(bending along the chosen diagonals.)

Have a Lag. torus fibration $\Phi_{\triangle} : \mathcal{M}_{\vec{r}} \to \mathbb{R}^n$

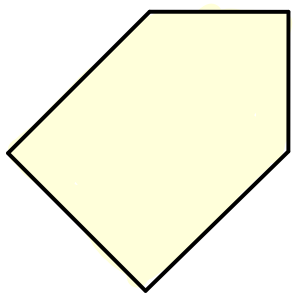
(given by the length of the chosen diagonals)

- The image of $\Phi_{\triangle}$ is a polytope given by triangle inequalities.

e.g> $n=2$, $\vec{r} = (1,1,1,1,1)$ $\Phi_{\triangle} = (\Phi_{13}, \Phi_{14})$



$\in \mathcal{M}_{\vec{r}}$ in $\mathbb{R}^3$



$=: \Delta_{\vec{r}} = \mathrm{Im}\, \Phi_{\triangle}$ in $\mathbb{R}^{n=2}$

SYZ mirrors of polygon spaces

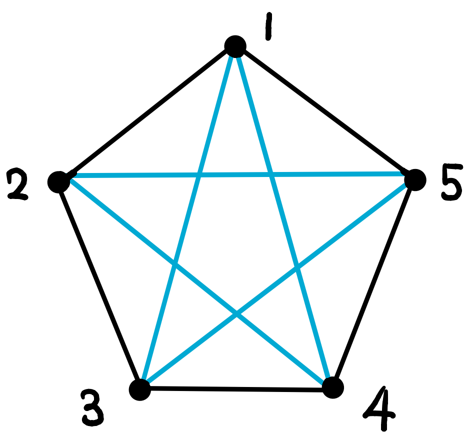
Theorem (Lau – K. – Zheng)

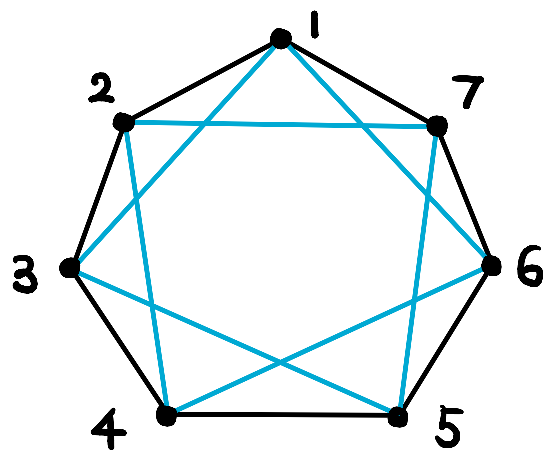
Assume that $\vec{r}$ is equilateral and generic. $X := \mathcal{M}_{\vec{r}}$ (Polygon space)

Then an SYZ mirror of X is a LG model $(\check{X}, W)$

- $\check{X} := Gr(2, \mathbb{C}^{n+3}) \cap \{p_{12} = p_{23} = \cdots = p_{n+2,n+3} = p_{1,n+3}\}$

  (Regard $Gr(2, \mathbb{C}^{n+3}) \subseteq \mathbb{P}(\Lambda^2 \mathbb{C}^{n+3})$ via the Plücker embedding $p_{i,j}$)
- $W : \check{X} \longrightarrow \mathbb{C}$ $\quad W(p) := \sum_{i=1}^{n+3} p_{i,i+2}$





Disk potential functions for caterpillar bending systems

Assume that $\vec{r}$ is equilateral and generic. $X := \mathcal{M}_{\vec{r}}$ (Polygon space)

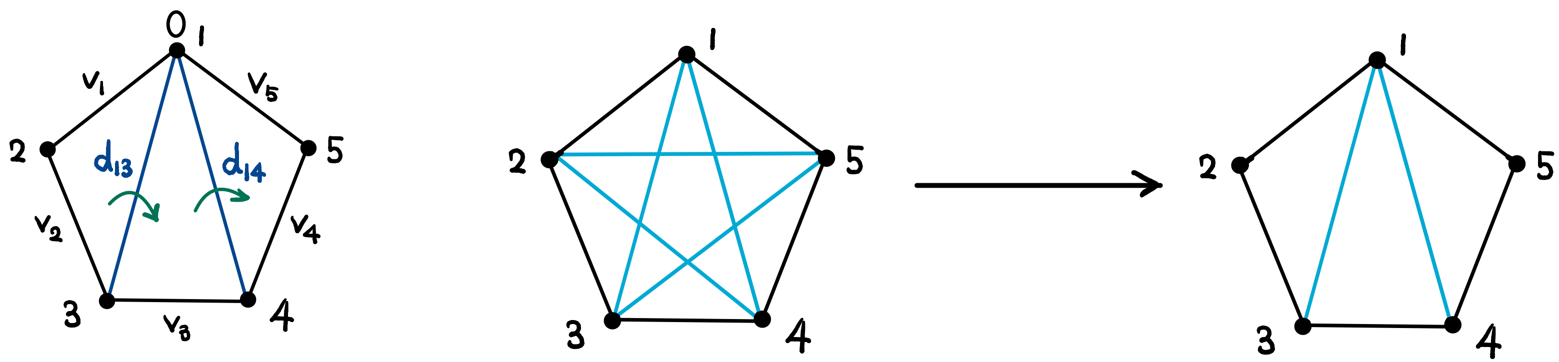
- $\Phi_{\triangle}$: the carterpillar bending system $\Phi_{\triangle} = (\Phi_{13}, \Phi_{14}, \cdots, \Phi_{1,n+2})$

$L_{\triangle} :=$ the monotone Lagrangian torus fiber of $\Phi_{\triangle}$, located at the center.

$$W_{L_{\triangle}} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{1,n+2} + \frac{2}{p_{1,n+2}}\right) + \sum_{j=3}^{n+1}\left(\frac{p_{1,j+1}}{p_{1,j}} + \frac{p_{1,j}}{p_{1,j+1}} + \frac{1}{p_{1,j}\,p_{1,j+1}}\right)$$

e.g> $n=2$, $\vec{r} = (1,1,1,1,1)$, $\mathcal{M}_{\vec{r}} \simeq dP_5$

$$W_{L_{\triangle}} = W\big|_{\mathbb{C}^*_{p_{13},p_{14}}} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{14} + \frac{2}{p_{14}}\right) + \left(\frac{p_{14}}{p_{13}} + \frac{p_{13}}{p_{14}} + \frac{1}{p_{13}p_{14}}\right)$$



<u>Remark</u> (FOOO & Pascaleff-Tonkonog) compute the disk potential of dP_5.

Cluster varieties via SYZ mirror symmetry

(Fomin – Zelevinsky)

- **Seed**

$$\vec{x} = (\underbrace{x_1, x_2, \cdots, x_n}_{\text{cluster var.}}, \underbrace{x_{n+1}, \cdots, x_m}_{\text{frozen var.}})$$

$$B = \left.\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}_{m\times n}\right. \quad \boxed{\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}} =: \tilde{B}$$

skew-symmetric (symmetrizable)

- **Mutations**

$$\mu_k(\vec{x}, B) = (\vec{x}', B') \quad (k = 1, 2, \cdots, n)$$

$$(*)\begin{cases} x_1' = x_1 \\ \quad\vdots \\ x_k' = x_k^{-1} \cdot \left(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \right) \\ \quad\vdots \\ x_m' = x_m \end{cases}$$

$$B' = (b_{ij}')_{m\times n}$$

$$b_{ij}' = \begin{cases} -b_{ij} & (i = k \text{ or } j = k) \\ b_{ij} \pm b_{ik} b_{kj} & (b_{ik} \,\&\, b_{kj} \gtrless 0) \\ b_{ij} & (\text{otherwise}) \end{cases}$$

(Fock – Goncharov) $\mathcal{U}_{\vec{x}} := \mathrm{Spec}(\mathbb{C}[\vec{x}^{\pm}]) \simeq (\mathbb{C}^*)^m$

$\mathcal{A} := \bigcup_{(\vec{x}, B)} \mathcal{U}_{\vec{x}}$ glue via the birational transformation $(*)$

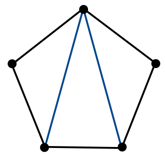
Q. For a given cluster variety X, find a symp. mfld and a Lag. torus fibration whose Floer theoretical mirror is X.

Polygon spaces – cluster varieties of type A

Symplectic	mirror	Complex
Polygon space, bending systems		Cluster variety of Cartan A type without frozen variables

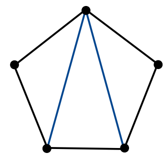
e.g.> $n=2$, $\vec{r}=(1,1,1,1,1)$, $\mathcal{M}_{\vec{r}} \simeq dP_5$

Initial seed $((x_1 = p_{13}, x_2 = p_{14}), B)$. Set $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B' = -B$.



$$W_{L} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{14} + \frac{2}{p_{14}}\right) + \left(\frac{p_{13}}{p_{14}} + \frac{p_{14}}{p_{13}} + \frac{1}{p_{13}p_{14}}\right)$$

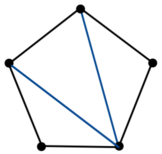
$((x_1, x_2), B) \sim ((x_7, x_6), B')$, $x_2 = x_7$, $x_1 = x_6$



$$W_{L} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{14} + \frac{2}{p_{14}}\right) + \left(\frac{p_{13}}{p_{14}} + \frac{p_{14}}{p_{13}} + \frac{1}{p_{13}p_{14}}\right)$$

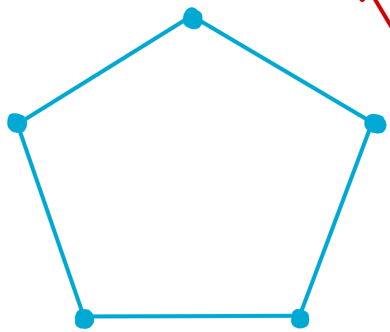
$((x_1, x_2), B) \xleftrightarrow{\mu_1} ((x_3, x_2), B')$

$((x_7, x_6), B') \xleftrightarrow{\mu_1} ((x_5, x_6), B)$

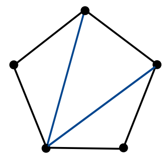


$$W_{L} = \left(p_{24} + \frac{2}{p_{24}}\right) + \left(p_{14} + \frac{2}{p_{14}}\right) + \left(\frac{p_{24}}{p_{14}} + \frac{p_{14}}{p_{24}} + \frac{1}{p_{14}p_{24}}\right)$$

$((x_3, x_2), B')$, $x_3 = \dfrac{x_2 + 1}{x_1}$



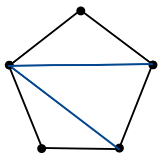
$((x_5, x_6), B)$, $x_6 = x_1$



$$W_{L} = \left(p_{13} + \frac{2}{p_{13}}\right) + \left(p_{35} + \frac{2}{p_{35}}\right) + \left(\frac{p_{13}}{p_{35}} + \frac{p_{35}}{p_{13}} + \frac{1}{p_{13}p_{35}}\right)$$

$((x_3, x_2), B') \xleftrightarrow{\mu_2} ((x_3, x_4), B)$

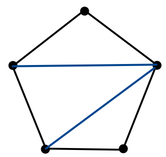
$((x_5, x_6), B) \xleftrightarrow{\mu_2} ((x_5, x_4), B')$



$$W_{L} = \left(p_{24} + \frac{2}{p_{24}}\right) + \left(p_{25} + \frac{2}{p_{25}}\right) + \left(\frac{p_{24}}{p_{25}} + \frac{p_{25}}{p_{24}} + \frac{1}{p_{24}p_{25}}\right)$$

$((x_3, x_4), B) \xleftrightarrow{\mu_1} ((x_5, x_4), B')$

$x_4 = \dfrac{x_1 + x_2 + 1}{x_1 x_2}$, $x_5 = \dfrac{x_1 + 1}{x_2}$



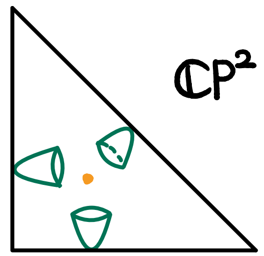
$$W_{L} = \left(p_{25} + \frac{2}{p_{25}}\right) + \left(p_{35} + \frac{2}{p_{35}}\right) + \left(\frac{p_{25}}{p_{35}} + \frac{p_{35}}{p_{25}} + \frac{1}{p_{25}p_{35}}\right)$$

Computation of disk potential functions

(Toric case)

Cho–Oh, FOOO, Woodward, Chan–Lau–Leung–Tseng...

e.g.> Fano toric case $\quad W_{\text{disk}} = W_{\text{GHV}}$



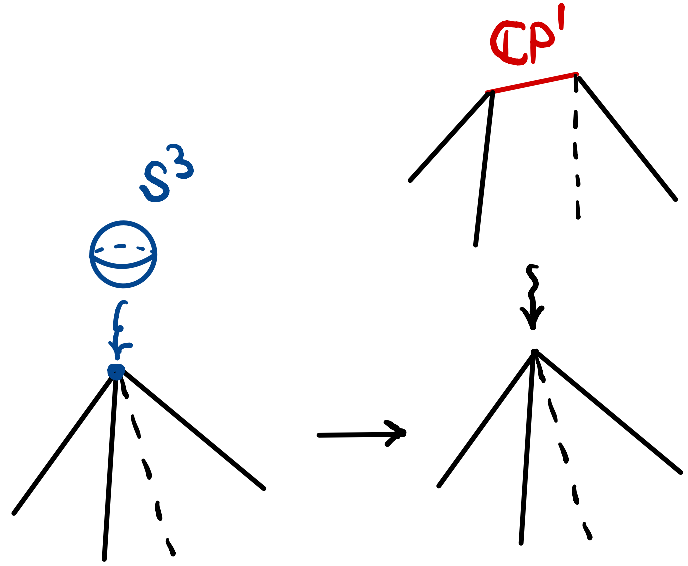
$$W = y_1 + y_2 + \frac{1}{y_1 y_2}$$

(Nishinou–Nohara–Ueda) concerns

Toric degenerations $\mathfrak{X} = \{\mathfrak{X}_t\}_{t\in\mathbb{C}} \to \mathbb{C}$ satisfying

⊛ each $\mathfrak{X}_t$ is Fano, $\mathfrak{X}_0$ has only conifold singularity

$$\begin{tikzcd} \mathfrak{X}_1 \arrow[rr, squiggly, "\varphi"] \arrow[dr, "\Phi_1"'] & & \mathfrak{X}_0 \arrow[dl, "\Phi_0"] \\ & \Delta & \end{tikzcd}$$



$\Rightarrow$ $W_{\text{disc}} = W_{\text{GHV}}$ (resemble the Fano toric case)

e.g.> Gelfand–Zeitlin systems of partial flag varieties of type A $\quad SL_n(\mathbb{C})/P$

Disk potential functions and toric degenerations

But. still far from understanding disk potentials of general toric degenerations.

Examples not satisfying ⊛ include

- Partial flag manifolds (other than type A/C)
  Gelfand – Zeitlin toric deg. of partial flag var. of type B and D.

- Even for partial flag var. of type A
  Toric deg. associated to string polytopes

$$
\begin{tikzcd}
 & & \mathcal{X}_1 \arrow[dll, squiggly] \arrow[ddl, squiggly] \arrow[ddr, squiggly] \arrow[drr, squiggly] & & \\
\mathcal{X}_0 \arrow[d] & & & & \mathcal{X}_0''' \arrow[d] \\
\Delta & \mathcal{X}_0' \arrow[d] & & \mathcal{X}_0'' \arrow[d] & \Delta''' \\
 & \Delta' & & \Delta'' &
\end{tikzcd}
$$

(Cho – K. – Lee – Park)

  - toric deg "close to" GZ case satisfies ⊛
  - found some toric deg. not satisfying ⊛

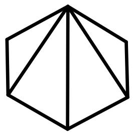
- 3D polygon spaces $\mathcal{M}_{\vec{r}}$

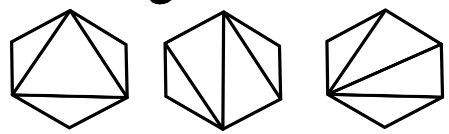
Need to enhance understanding beyond ⊛

Generalized Gelfand–Zeitlin systems and Marsh–Rietsch mirrors

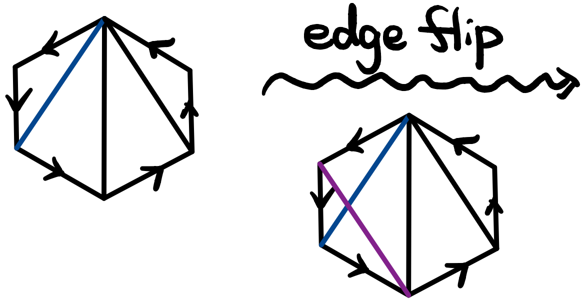
(Hausmann–Knutson, Nohara–Ueda)

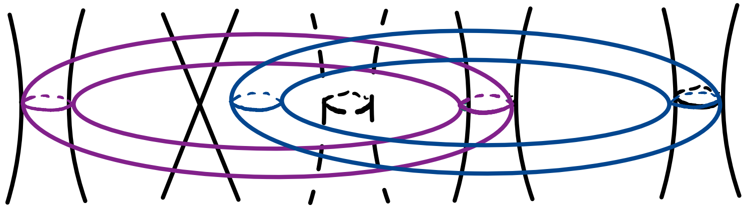
$Gr(2, \mathbb{C}^{n+3})$	Gelfand–Zeitlin system	Generalized GZ systems
$\mathcal{M}_{\vec{r}}$	Caterpillar bending system	Bending systems



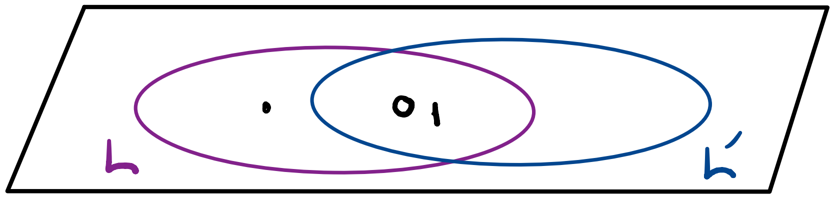


(Nohara–Ueda) explored relation between GGZ systems and recovered MR mirror.





$\times (\mathbb{C}^*)^{2n}$



- $\Phi = (\Phi_{13}, \Phi_{14}, \Phi_{15}, \Phi_{T_{U(n+3)}})$
- $\Phi = (\Phi_{24}, \Phi_{14}, \Phi_{15}, \Phi_{T_{U(n+3)}})$

(Marsh–Rietsch) derived a LG mirror whose Jacobian ring is isom. to QH of Gr.

$$\check{X} = Gr(2, \mathbb{C}^{n+3}) \setminus D(\text{proj. Richardson}),\quad W_q := \sum \frac{p_{i,i+2}}{p_{i,i+1}} : \check{X} \longrightarrow \mathbb{C}$$

Frozen and cluster variables as holonomy variables

Symplectic

$Gr(2, \mathbb{C}^{n+3})$, $L_{\triangle}$: torus orbit

$\Phi_{d_1}, \cdots, \Phi_{d_n}$: local torus act.

$\Phi_{e_1}, \cdots, \Phi_{e_{n+2}}$: global torus act.

Choose loops as orbits

$\theta_1, \cdots, \theta_n$; $\theta_{n+1}, \cdots, \theta_{2n+2}$

$(L_{\triangle}, \nabla) \sim (L_{\triangle'}, \nabla')$

$Gr(2, \mathbb{C}^{n+3})$ $L_{\triangle}, L_{\triangle'} \cdots$

$\big\downarrow /\!/ T_{U(n+3)}$

$\mathcal{M}_{\vec{r}}$, $\underline{L}_{\triangle}, \underline{L}_{\triangle'} \cdots$

Complex

$\check{U}_{\triangle} \simeq (\mathbb{C}^*)^{2n+2}$

$p_{1,i+2} = \mathrm{hol}_{\nabla}(\theta_i)$ $(i = 1, \cdots, n)$

(cluster variables)

$p_{i-n, i+1-n} = \mathrm{hol}_{\nabla}(\theta_i)$ $(i = n+1, \cdots, 2n+2)$

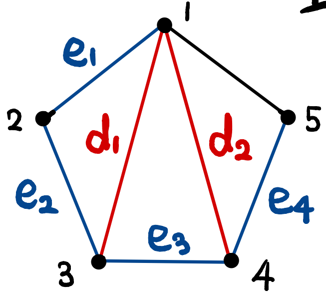
(frozen variables)

Wall-crossing = mutation

$\coprod \check{U}_{\triangle}$ $(\simeq (\mathbb{C}^*)^{2n+2})$, $B_{(2n+2)\times n}$

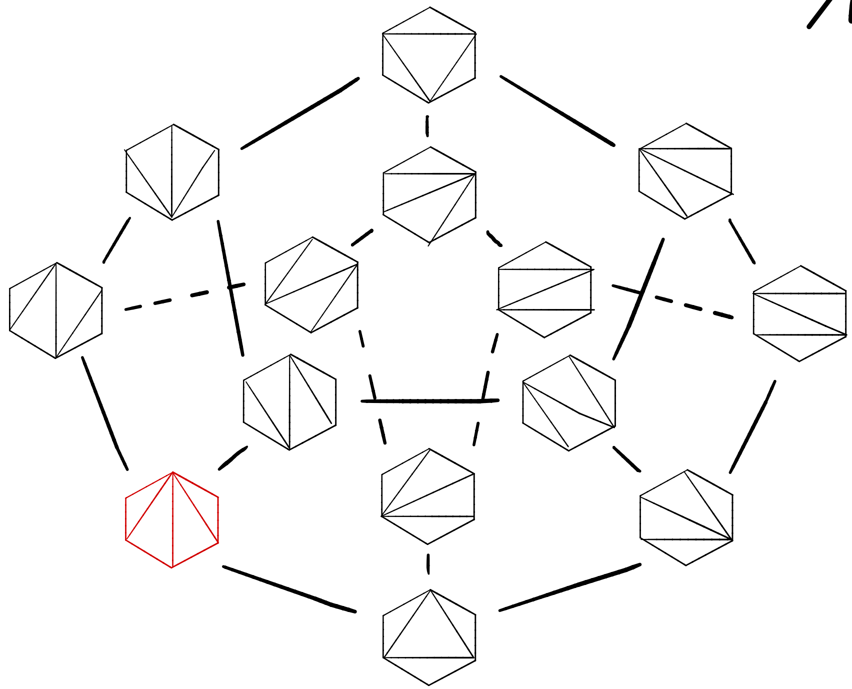
$\big\downarrow$ $p_{12} = p_{23} = \cdots = p_{n+2,n+3} = p_{1,n+3}$

$\coprod \check{\underline{U}}_{\triangle}$ $(\simeq (\mathbb{C}^*)^{n})$, $\widetilde{B}_{n\times n}$



Strategy

- To calculate the disk potential functions for various bending systems, it suffices to compute one of them.



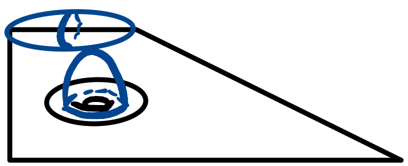
- Global ⇒ Local ⇒ new Global.

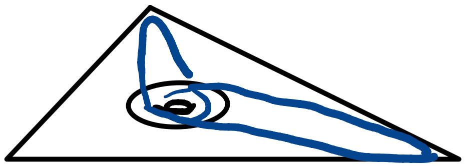
Global Partial flag varieties of various types (Cho – K. – Oh)

Local $T^*V_k(\mathbb{C}^n)$, $T^*V_k(\mathbb{R}^n)$, $T^*U(n)$, $T^*SU(n)$, $T^*SO(n)$, T^*S^n

New Global Polygon spaces.

⚠ Each disk corresponding to a term in W is contained in a local model.





Singular fibers of bending systems

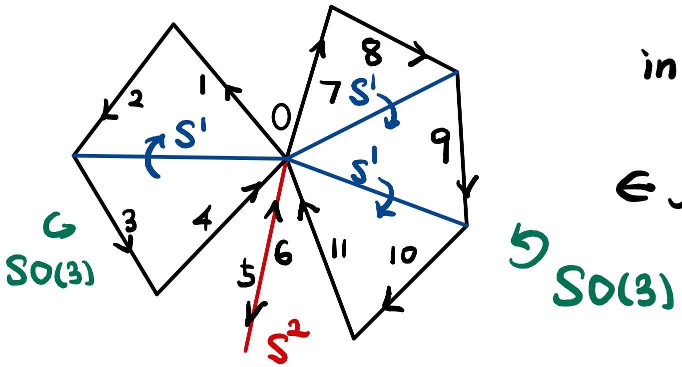
(Bouloc) If $\vec{r}$ is generic, the fiber of any point under $\Phi_{\vec{r}}$ is an isotropic submfld diffeomorphic to

$$SO(3)^{m_1} \times (S^2)^{m_2} \times T^{m_3}$$

Need local models. $T^*SO(3)$, T^*S^2

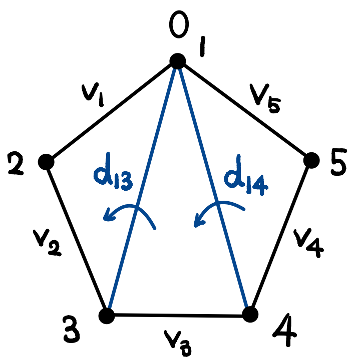
e.g> $n=8$ $\vec{r} = (1, 1, \cdots, 1)$

$(|d_{13}| > 0,\ |d_{14}| = 1,\ |d_{15}| = 0,\ |d_{16}| = 1,\ |d_{17}| = 0,\ |d_{18}| = 1,\ |d_{19}| > 0,\ |d_{20}| > 0)$ given

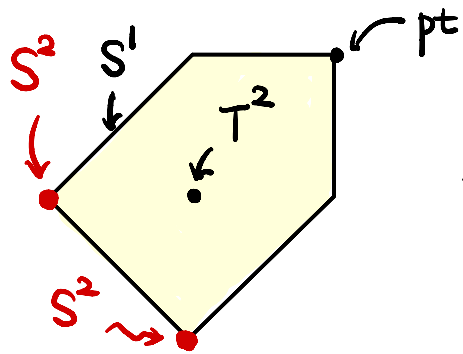


$\in \mathcal{M}_{\vec{r}}$ in $\mathbb{R}^3$ $\rightsquigarrow$ The fiber is $\approx SO(3)^{2-1} \times S^2 \times T^3$

e.g> $n=2$, $\vec{r} = (1,1,1,1,1)$, $\mathcal{M}_{\vec{r}} \simeq dP_5$



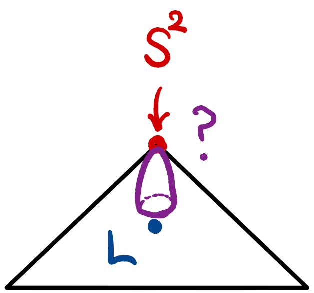
$\in \mathcal{M}_{\vec{r}}$ in $\mathbb{R}^3$



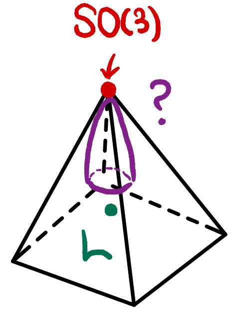
$= \mathrm{Im}\, \Phi_{\triangle}$ in $\mathbb{R}^2$

Computation of disk potential functions for $OG(1,\mathbb{C}^4)$ and $OG(2,\mathbb{C}^5)$

$OG(1,\mathbb{C}^4)$



$OG(2,\mathbb{C}^5)$



$$W_L(x) = x_2 + x_1x_2 + \frac{x_2}{x_1} + a\frac{1}{x_2}$$

$$W_L(x) = x_3 + \frac{1}{x_1x_3} + \frac{1}{x_2x_3} + \frac{x_1}{x_3} + \frac{x_2}{x_3} + b\frac{1}{x_3}$$

- No matter what $a, b\ (\in \mathbb{Z})$ are, L admits ∇ s.t. (L, ∇) is a nonzero obj.
- (Sheridan, AFOOO) $c_1(TOG) \cup - \curvearrowright QH^\bullet(OG)$

$Fuk_\lambda(OG(1,\mathbb{C}^4))$ is non-trivial when $\lambda = 4, 0, -4$ $(\Leftarrow OG(1,\mathbb{C}^4) \simeq \mathbb{CP}^1 \times \mathbb{CP}^1)$

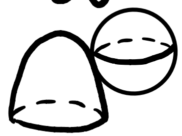
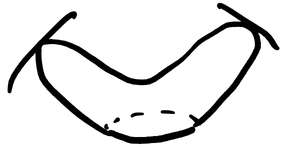
$Fuk_\lambda(OG(2,\mathbb{C}^5))$ is non-trivial when $\lambda = 4, -4, 4i, -4i$ $(\Leftarrow OG(2,\mathbb{C}^5) \simeq \mathbb{CP}^3)$

$\Rightarrow a = 2$ & $b = 0$

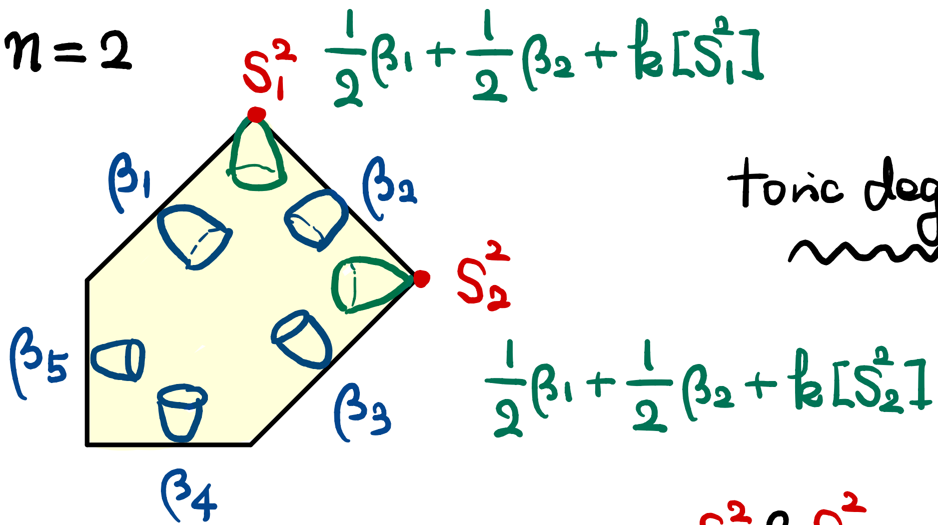
In sum,

$$W_L(x) = x_2 + x_1x_2 + \frac{x_2}{x_1} + 2\cdot\frac{1}{x_2} \qquad W_L(x) = x_3 + \frac{1}{x_1x_3} + \frac{1}{x_2x_3} + \frac{x_1}{x_3} + \frac{x_2}{x_3}$$

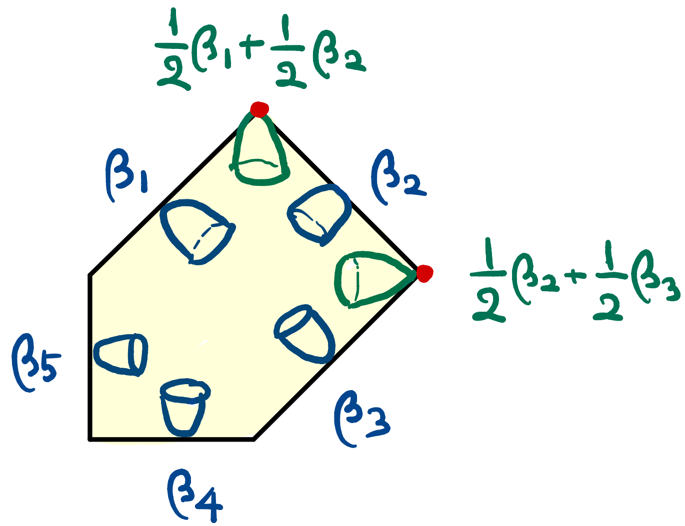
Computation of disk potential functions for polygon spaces $\mathcal{M}_{\vec{r}}$

- For a generic $\vec{r}$, $\mathcal{M}_{\vec{r}}$ is symp. Fano $\Rightarrow$  (X)
- Area $= 1$ + $\partial\beta \in N$ determine effective homotopy classes of Maslov index 2 $\Rightarrow$  (X)
- Local models compute open GW invariants

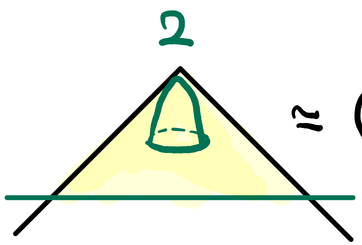
e.g.) $n=2$



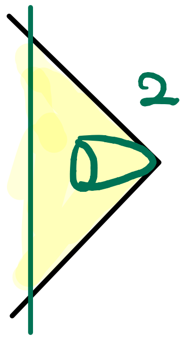
toric degeneration



$$\pi_2(\mathcal{M}_{\vec{r}}, L_1) \otimes \mathbb{Q} \simeq \mathbb{Q}^7 \xrightarrow{S_1^2 \,\&\, S_2^2 \text{ collapse}} \pi_2(X_0, L_0) \otimes \mathbb{Q} \simeq \mathbb{Q}^5$$



$\simeq OG(1, \mathbb{C}^4) \simeq$



$$\bigcup_k \mathcal{M}_1\left(L_1, \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2 + k[S_i^2]\right) \longrightarrow L_1$$

is of degree 2.

Computation of disk potential functions for polygon spaces $\mathcal{M}_{\vec{r}}$ (cont.)

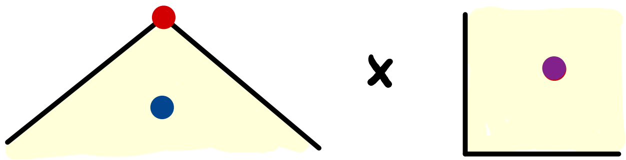
For higher dimensional cases,

e.g> $n = 4$. $\vec{r} = (1, 1, \cdots, 1)$.

In this case, need to care the following Lagrangian fibers

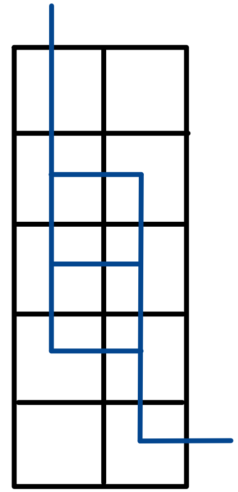
(Two $SO(3) \times T^1$
  Two $S^2 \times T^2$

For instance, $S^2 \times T^2$ occurs at $(|d_{13}|, |d_{13}|, |d_{13}|, |d_{13}|) = (0, 1, 1, 1)$



7

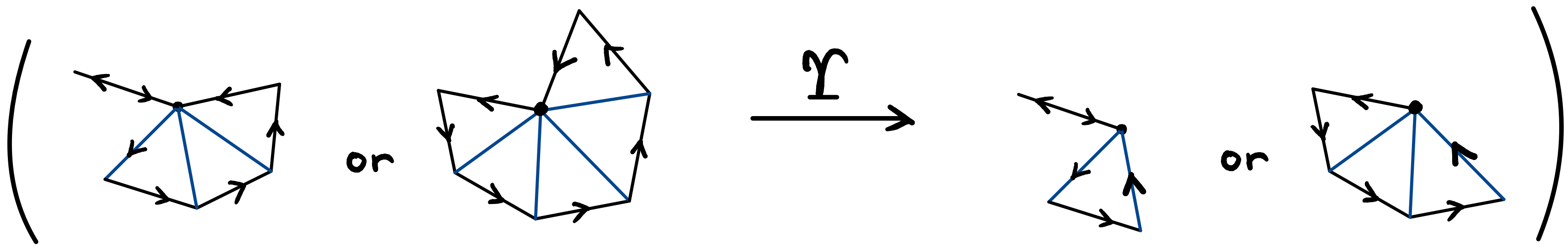
$10-u_4$	5
$8-u_3$	u_4
$6-u_2$	u_3
$4-u_1$	u_2
2	u_1



A nbd $\mathcal{U}$ around $S^2 \times T^2$ can be trivialized as $\mathcal{U}/\!\!/T^2 \times (\mathbb{C}^*)^2$ via toric deg.

Then have an orientation preserving diffeom.

$$\mathcal{M}(\mathcal{U}, T^4; \beta) \xrightarrow{\sim} \mathcal{M}(\mathcal{U}/\!\!/T^2, T^2; \underline{\beta}) \times T^2$$



Choice of $\vec{r}$

We are concerned with the equilateral case.

Q. What if we change $\vec{r}$?

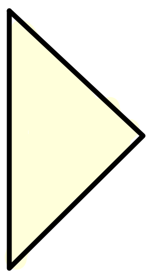
Can have different algebraic varieties / different fibrations

e.g> $\vec{r}_1 = (1.1.3.3.3)$ $\quad \mathcal{M}_{\vec{r}_1} \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$ $\quad W_1 = y_1 + \frac{y_2}{y_1} + \frac{2}{y_1} + \frac{1}{y_1 y_2}$

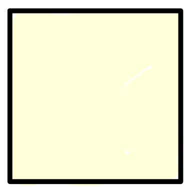
e.g> $\vec{r}_2 = (1.3.3.3.1)$ $\quad \mathcal{M}_{\vec{r}_2} \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$ $\quad W_2 = y_1 + \frac{1}{y_1} + y_2 + \frac{1}{y_2}$

e.g> $\vec{r}_3 = (1.1.3.2.2)$ $\quad \mathcal{M}_{\vec{r}_3} \simeq \mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ $\quad W_3 = y_1 + \frac{y_2}{y_1} + \frac{2}{y_1} + \frac{1}{y_1 y_2} + y_2$

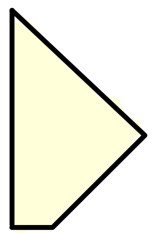
e.g> $\vec{r}_4 = (2.2.2.4.4)$ $\quad \mathcal{M}_{\vec{r}_4} \simeq \mathbb{C}P^2 \# 3\overline{\mathbb{C}P}^2$ $\quad W_4 = y_1 + \frac{y_2}{y_1} + \frac{1}{y_1 y_2} + \frac{2}{y_2} + \frac{y_1}{y_2}$



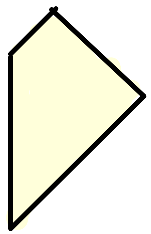
$\Delta_{\vec{r}_1}$



$\Delta_{\vec{r}_2}$



$\Delta_{\vec{r}_3}$



$\Delta_{\vec{r}_4}$

As long as $\mathcal{M}_{\vec{r}}$ is monotone. one can similarly compute W.

Thank You!